A Stanley-Elder Type Relationship for Overpartitions

Special Session on Early Career Number Theory Research with Combinatorics, Modular Forms, and Basic Hypergeometric Series

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Overview

- Preliminaries
- The Partition Case
- Lifting to Overpartitions

Definition

A *partition* π is a non-increasing sequence of positive integers. If the sum of these integers is n, then we write $\pi \vdash n$, or $|\pi| = n$. Let $p(n)$ denote the number of partitions of n.

The partitions $\pi \vdash 4$ are

$$\begin{array}{ll} (4) & (3,1) \\ (2,2) & (2,1,1) \\ (1,1,1,1). & \end{array}$$

Thus, $p(4) = 5$.

Preliminaries

Theorem (Stanley, Elder)

For each $j \geq 1$ the number of j's used in the partitions of n equals the number of parts which occur at least j times in a given partition of n, summed over all the partitions of n.

Again, consider $n = 4$.

$$\begin{array}{ll} (4) & (3,1) \\ (2,2) & (2,1,1) \\ (1,1,1,1) & \end{array}$$

There are three 2's, and three different occurences of 2 or more repeated parts in a single partition.

Obligatory Ramanujan Congruences Slide

Theorem (Ramanujan, Hardy; 1920)

For all $n \geq 0$,

$$p(5n+4) \equiv 0 \pmod{5}$$
$$p(7n+5) \equiv 0 \pmod{7}$$
$$p(11n+6) \equiv 0 \pmod{11}.$$

Preliminaries

Definition

The *rank* of π is equal to the largest part of π minus the number of parts of π.

For example,

$$r((4,4,1)) = 4 - 3 = 1.$$

Divvying the partitions $\pi \vdash (5n+4)$ according to their rank modulo 5 produces five sets of equal size. This technique also proves the modulo 7 congruence, but fails for the modulo 11 congruence.

Preliminaries

Definition

If a partition π does not contain any 1s, then the *crank* of π is defined to be the largest part of π.

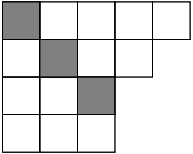
Otherwise, let $w(\pi)$ denote the number of 1's occurring in π, and let $\mu(\pi)$ denote the number of parts of π which are larger than $w(\pi)$. In this case, the crank of π is defined to be

$$c(\pi) = \mu(\pi) - w(\pi).$$

Payoff: The crank proves all three Ramanujan congruences.

The Partition Case

Recall, the *Frobenius symbol* is a $2 \times k$ array which enumerates the number of boxes to the right of the main diagonal of a Young diagram, and then the number of boxes below the main diagonal.



$$(5,4,3,3) \leftrightarrow \begin{pmatrix} 4 & 2 & 0 \\ 3 & 2 & 1 \end{pmatrix}$$

The Partition Case

Theorem (Andrews, Dastidar, M., 2021)

For each $j \geq 0$, the number of partitions of n with cranks $> j$ equals one half of the number of j's occuring in the Frobenius symbols for the partitions of n.

This is a particuarly surprising result, because the Frobenius symbol is much more friendly to the rank funciton than the crank.

The Partition Case

Theorem (Andrews, Dastidar, M., 2021)

Let π be a partition of n with $c(\pi) = k > 0$. Then there is a one-to-one correspondence between π and a set consisting of two occurrences of each of the integers i with $0 \leq i \leq k-1$ among all of the parts of the Frobenius symbols for the partitions of n.

Corollary

The sum of the side lengths of all the Durfee squares in the partitions of n equals the sum of all the positive cranks in the partitions of n. Further,

$$\frac{1}{2}M_2(n) = np(n).$$

Lifting to Overpartitions

Definition

An *overpartition* is a non-increasing sequence of positive integers, where the first occurrence of each part may be overlined.

The overpartitions $\pi \vdash 3$ are

$$\begin{array}{llll} (3) & (\overline{3}) & (1,1,1) & (\overline{1},1,1) \\ (2,1) & (2,\overline{1}) & (\overline{2},1) & (\overline{2},\overline{1}). \end{array}$$

Overpartitions share many of the features that lead to the partition result: Frobenius representations, ranks, and cranks.

Lifting to Overpartitions

Definition

For $k \geq 1$, the kth residual partition of π is a partition π' consisting of $1/k$th of each of the non-overlined parts of π that are divisible by k. The kth *residual crank* of π is then defined to be $c_k(\pi) = c(\pi')$.

For example,

$$\begin{aligned} c_1((4,\overline{3},2)) &= c((4,2)) = 4 \\ c_2((4,\overline{3},2)) &= c((2,1)) = 0 \\ c_3((4,\overline{3},2)) &= c(\emptyset) = 0 \\ c_4((4,\overline{3},2)) &= c((1)) = -1 \\ c_k((4,\overline{3},2)) &= c(\emptyset) = 0, \text{ for } k \geq 5. \end{aligned}$$

Lifting to Overpartitions

Let $k = 1$. We consider the first residual crank in relation to the first Frobenius representation of overpartitions.

Theorem (Corteel, Lovejoy; 2004)

There is a bijection between overpartitions π and generalized Frobenius representations $\nu = (\alpha, \beta)^T$ where α is a partition into distinct parts and β is an overpartition into nonnegative parts such that $|\lambda| = |\nu|$.

For example,

$$(3,3,3,3,3,\overline{2}) \leftrightarrow \begin{pmatrix} 3 & 2 & 1 \\ \overline{4} & 4 & \overline{3} \end{pmatrix}.$$

Sketch

Consider an overpartition as a vector partition (μ, λ), where μ consists of all the overlined parts, and λ consists of all the nonoverlined parts. Note that the first residual crank ignores all parts of μ.

Tracking the parts of λ through Corteel and Lovejoy's map gives a simillar bijection as in the partition case. (Frobenius symbols of ordinary partitions coincide with first Frobenius representations with all parts of β overlined.)

Future Study

For $k = 2$, we expect a simillar result to hold between the second residual crank and the second Frobenius representation of overpartitions. The map between overpartitions and second Frobenius representations is somewhat opaque.

For $k \geq 3$, we are not aware of other Frobenius representations to attempt such a comparisson. However, this idea may well be applicable to other crank-like functions defined on vector partitions.

Thank you!